DYNAMIC PROGRAMMING FOR A MARKOV–SWITCHING JUMP-DIFFUSION

N. AZEVEDO, D. PINHEIRO, AND G.-W. WEBER

Abstract. We prove a dynamic programming principle for a finite horizon optimal control problem for which the state variable dynamics are defined by a Markov-switching jump-diffusion stochastic differential equation. As an application of our results, we study a finite horizon consumption-investment problem in a jump-diffusion financial market consisting of one risk-free asset and one risky asset whose coefficients are assumed to depend on the state of a continuous time finite state Markov process. We provide a detailed study of the optimal strategies for this problem, for both power utilities and logarithmic utilities.

Keywords: Stochastic Optimal Control, Jump-diffusion, Markov-switching, Optimal consumption-investment.
AMS classification: 93E20, 49L20, 91G10

1. Introduction

Dynamic programming was firstly introduced by Richard Bellman in the 1950s to deal with calculus of variations and optimal control problems [2, 3, 4, 5]. Further developments have been obtained since then by a number of scholars including Florentin [9, 10] and Kushner [16] among others. The approach introduced by Bellman relies on the description of the value function associated with a given optimal control problem through a backwards recursive relation, know currently as Bellman's optimality principle. Under additional regularity conditions, it can be proved that such value function is also the solution of a partial differential equation, known as Hamilton-Jacobi-Bellman equation. A very complete treatment of the modern theory of optimal control problems can be found on the excellent monographs by Fleming and Soner [8], Yong and Zhou [23] and Oksendal and Sulem [19].

In this paper, we consider an optimal control problem with finite deterministic horizon for which the state variables dynamics are defined by a Markov-switching jump-diffusion stochastic differential equation. We generalize Bellman's dynamic programming principle to this broader setup. Moreover, we provide conditions under which the corresponding value function is the solution of a Hamilton-Jacobi-Bellman equation, which in our case is a partial integro-differential equation due to the extra terms arising from the Lévy process terms and the Markov process driving the switching.

As an example of application of our results to Mathematical Finance, we consider Merton's consumption-investment problem in the setup of Markov-switching jump-diffusion financial markets. The consumption-investment problem was firstly studied by Merton in his seminal papers [17, 18]. This problem has been thoroughly studied ever since, including extensions to

jump-diffusion financial markets (see, e.g., the series of papers by Framstad et al. [11, 12, 13] and references therein). In what concerns Markov–switching behaviour in economics and finance, this has been considered by Hamilton in [14] to explain shifts in growth rates of GNP, Elliott et al. in [6, 7] to address problems related with option pricing and risk minimization, Zhang in [24] to determine optimal selling rules and by Zhang and Yin to deal with optimal asset allocation rules [25]. To the best of our knowledge, dynamic programming techniques have not yet been applied to the consumption-investment problem with an underlying Markov-switching jump-diffusion financial market.

This paper is organized as follows. In Section 2, we describe the setting we work with and formulate the problem we propose to address. Section 3 contains the dynamic programming principle and the HJB partial integro-differential equations, as well as the corresponding verification theorem and its proof. In Section 4, we address a consumption-investment problem and study the particular case of power utility functions and logarithmic utility functions. We conclude in Section 5.

2. Setup and problem formulation

Let $T > 0$ be a deterministic finite horizon and let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0,T]\}$ satisfying the usual conditions, i.e., $\mathbb{F}$ is an increasing, right-continuous filtration and $\mathcal{F}_0$ contains all P-null sets. For each $d \in \mathbb{N}$, let $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$ and let $\mathcal{B}_0^d$ be the Borel σ-field generated by the open subsets O of $\mathbb{R}_0^d$ whose closure does not contain 0.

We will consider the following stochastic processes throughout this paper:

(i) a standard M-dimensional Brownian motion $W(t) = \{W(t) : t \in [0,T]\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

(ii) a continuous time Markov process $\{\alpha(t) : t \in [0,T]\}$ with a finite space state $S = \{a_1, \ldots, a_n\}$ and generator $Q = (q_{ij})_{i,j \in S}$. Let $N_{ij}(t)$ denote the counting process given by

$$N_{ij}(t) = \sum_{0<s\leq t} I_{\{\alpha(s_-)=i\}} I_{\{\alpha(s)=j\}},$$

where I_A denotes the indicator function of a set A. Note that $N_{ij}(t)$ gives the number of jumps of the Markov process α from state i to state j up to time t. Define the intensity process by

$$\lambda_{ij}(t) = q_{ij} I_{\{\alpha(t_-)=i\}},$$

and introduce the martingale process $M_{ij}(t)$ given by

$$M_{ij}(t) = N_{ij}(t) - \int_0^t \lambda_{ij}(s)\mathrm{d}s.$$

The process $M_{ij}(t)$ is a purely discontinuous, square-integrable martingale which is null at the origin (see, e.g., [15, 21]).

(iii) a K-dimensional Lévy process $\{\eta(t) : t \in [0,T]\}$ with Poisson random measure $J(t,A)$ with intensity (or Lévy measure)

$$\nu(A) = E[J(1,A)].$$

Recall that for each $t > 0$, $\omega \in \Omega$, $J(t,\cdot)(\omega)$ is a counting measure on $\mathcal{B}_0^K$, and that for each $A \in \mathcal{B}_0^K$, $\{J(t,A) : t \in [0,T]\}$ is a Poisson process with intensity $\nu(A)$. For each $t \in [0,T]$ and $A \in \mathcal{B}_0^K$, define the compensated Poisson random measure of $\eta(\cdot)$ by

$$\tilde{J}(t,A) = J(t,A) - t\nu(A)$$

and notice that $\{\tilde{J}(t,A) : t \in [0,T]\}$ is a martingale-valued measure [1]. Finally, notice that

$$\begin{aligned}\tilde{J}(\mathrm{d}t,\mathrm{d}z) &= (\tilde{J}_1(\mathrm{d}t,\mathrm{d}z_1),\ldots,\tilde{J}_K(\mathrm{d}t,\mathrm{d}z_K))\\ &= (J_1(\mathrm{d}t,\mathrm{d}z_1)-\nu_1(\mathrm{d}z_1)\mathrm{d}t,\ldots,J_K(\mathrm{d}t,\mathrm{d}z_K)-\nu_K(\mathrm{d}z_K)\mathrm{d}t),\end{aligned}$$

where J_k, $k = 1,\ldots,K$, are independent Poisson random measures with Lévy measures ν_k coming from K independent (1-dimensional) Lévy processes $\eta_1,\ldots,\eta_K$.

We assume that the Brownian motion $W(\cdot)$, the Markov process $\alpha(\cdot)$ and the Lévy process $\eta(\cdot)$ are all independent and adapted to the filtration $\mathbb{F}$.

For each $n \in \mathbb{N}$, denote by $L^1_{\mathcal{F}_t}([0,T];\mathbb{R}^n)$ the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes $x(\cdot)$ such that

$$E\left[\int_0^T \|x(t)\|\ \mathrm{d}t\right] < \infty$$

and by $L^1_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ the set of $\mathbb{R}^n$-valued $\mathcal{F}_T$-measurable random variables Y such that $E[\|Y\|]$ is finite, where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. The spaces $L^1_{\mathcal{F}_t}([0,T];\mathbb{R}^n)$ and $L^1_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ are defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$.

We introduce the following technical assumptions:

(A1) (U,d) is a Polish metric space, i.e., a complete separable metric space.

(A2) The maps $f : [0,T]\times\mathbb{R}^N\times S\times U \to \mathbb{R}^N$, $g : [0,T]\times\mathbb{R}^N\times S\times U \to \mathbb{R}^{N\times M}$, $h : [0,T]\times\mathbb{R}^N\times S\times U\times\mathbb{R}_0^K \to \mathbb{R}^{N\times K}$, $\Psi : [0,T]\times\mathbb{R}^N\times S \to \mathbb{R}$ and $L : [0,T]\times\mathbb{R}^N\times S\times U \to \mathbb{R}$, are such that:
(i) for each fixed $a \in S$, $f(\cdot,\cdot,a,\cdot)$, $g(\cdot,\cdot,a,\cdot)$, $h(\cdot,\cdot,a,\cdot,\cdot)$, $\Psi(\cdot,\cdot,a)$ and $L(\cdot,\cdot,\cdot,a)$ are uniformly continuous;
(ii) each column $h^{(k)}$ of the $N\times K$ matrix $h(t,x,a,u,z) = [h_{ik}]$ depends on z only through the k^{th} coordinate z_k, i.e.,

$$h^{(k)}(t,x,a,u,z) = h^{(k)}(t,x,a,u,z_k), \quad z = (z_1,\ldots,z_K) \in \mathbb{R}_0^K;$$

(iii) for each fixed $a \in S$, there exists $C > 0$ such that for $\varphi(t,x,u) = f(t,x,a,u), g(t,x,a,u), \Psi(t,x,a), L(t,x,a,u)$, we have

$$\begin{aligned}&|\varphi(t,x,u)-\varphi(t,y,u)|^2 < C|x-y|^2,\\ &|\varphi(t,0,u)|^2 < C,\end{aligned}$$

and for each $k = 1,\ldots,K$, we have

$$\int_{\mathbb{R}_0^1}\left|h^{(k)}(t,x,a,u,z_k)-h^{(k)}(t,y,a,u,z_k)\right|^2\nu_k(\mathrm{d}z_k) < C|x-y|^2,$$

$$\int_{\mathbb{R}_0^1}\left|h^{(k)}(t,0,a,u,z_k)\right|^2\nu_k(\mathrm{d}z_k) < C.$$

We consider a stochastic controlled system of the form

$$\begin{aligned} \mathrm{d}X(t) &= f\left(t, X(t_-), \alpha(t_-), u(t_-)\right)\mathrm{d}t + g\left(t, X(t_-), \alpha(t_-), u(t_-)\right)\mathrm{d}W(t) \\ &\quad + \int_{\mathbb{R}_0^K} h\left(t, X(t_-), \alpha(t_-), u(t_-), z\right)\tilde{J}(\mathrm{d}t, \mathrm{d}z) \quad (t \in [0,T]), \qquad (1) \\ X(0) &= x, \quad \alpha(0) = a, \end{aligned}$$

together with an objective functional of the form

$$J(x,a;u(\cdot)) = E\Bigg[\int_0^T L\left(t, X_{0,x,a}(t;u(\cdot)), \alpha_{0,a}(t), u(t)\right)\mathrm{d}t + \Psi\left(T, X_{0,x,a}(T;u(\cdot)), \alpha_{0,a}(T)\right)\Bigg], \qquad (2)$$

where $(X_{0,x,a}(t;u(\cdot)), \alpha_{0,a}(t)) \in \mathbb{R}^N \times S$ denotes the state trajectory associated with a control trajectory $u(\cdot)$ and starting from (x,a) when $t = 0$.

Note that the components of Eqn. (1) take the form

$$\begin{aligned} \mathrm{d}X_i(t) &= f_i\left(t, X(t_-), \alpha(t_-), u(t_-)\right)\mathrm{d}t \\ &\quad + \sum_{j=1}^{M} g_{ij}\left(t, X(t_-), \alpha(t_-), u(t_-)\right)\mathrm{d}W_j(t) \\ &\quad + \sum_{k=1}^{K}\int_{\mathbb{R}_0^1} h_{ik}\left(t, X(t_-), \alpha(t_-), u(t_-), z_k\right)\tilde{J}_k(\mathrm{d}t, \mathrm{d}z_k), \end{aligned}$$

where f_i denotes the i^{th} component of f, g_{ij} denotes the (i,j) entry of matrix g, and h_{ik} denotes the (i,k) entry of matrix h.

We say that the control process $u : [0,T] \times \Omega \to U$ is a *strong admissible control* if u is measurable and $\{\mathcal{F}_t\}$-adapted, the stochastic differential equation (1) has a unique strong solution and

$$E\left[\int_0^T \left|L\left(t, X_{0,x,a}(t;u(\cdot)), \alpha_{0,a}(t), u(t)\right)\right|\mathrm{d}t\right] < \infty,$$

$$E\left[\left|\Psi\left(T, X_{0,x,a}(T;u(\cdot)), \alpha_{0,a}(T)\right)\right|\right] < \infty.$$

We denote the set of all strong admissible controls by $\mathcal{U}^s[0,T]$.

The stochastic optimal control problem is to find a control $u \in \mathcal{U}^s[0,T]$ which maximizes the objective functional $J(x,a;u(\cdot))$ subject to the state equation (1) over the set of admissible controls $\mathcal{U}^s[0,T]$. We will use dynamic programming techniques to address the maximization problem described above. In order to proceed, we need to consider the weak formulation of the stochastic control problem under consideration as an auxiliary tool.

For any $(s,y,i) \in [0,T) \times \mathbb{R}^N \times S$, consider the state equation:

$$\begin{aligned} \mathrm{d}X(t) &= f\left(t, X(t_-), \alpha(t_-), u(t_-)\right)\mathrm{d}t + g\left(t, X(t_-), \alpha(t_-), u(t_-)\right)\mathrm{d}W(t) \\ &\quad + \int_{\mathbb{R}_0^K} h\left(t, X(t_-), \alpha(t_-), u(t_-), z\right)\tilde{J}(\mathrm{d}t, \mathrm{d}z) \quad (t \in [s,T]), \qquad (3) \\ X(s) &= y, \quad \alpha(s) = i, \end{aligned}$$

along with the objective functional

$$J(s,y,i;u(\cdot)) = E\Bigg[\int_s^T L\left(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t)\right)\mathrm{d}t \\ +\Psi\left(T, X_{s,y,i}(T;u(\cdot)), \alpha_{s,i}(T)\right)\Bigg], \quad (4)$$

where $(X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t)) \in \mathbb{R}^N \times S$ is the solution of Eqn. (3) associated with the control $u(\cdot)$ and starting from (y,i) when $t = s$.

For each $s \in [0,T)$, we denote by $\mathcal{U}^w[s,T]$ the set of 8-tuples

$$(\Omega, \mathcal{F}, \mathbb{F}, P, W(\cdot), \alpha(\cdot), \eta(\cdot), u(\cdot))$$

for which the following conditions hold:

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space;
(ii) $\mathbb{F} = \{\mathcal{F}_t^s\}_{t\geq s}$ is a right-continuous filtration;
(iii) $\{W(t) : t \in [s,T]\}$ is a M-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ over $[s,T]$ and adapted to the filtration $\mathbb{F}$;
(iv) $\{\alpha(t) : t \in [s,T]\}$ is a continuous time Markov process on $(\Omega, \mathcal{F}, P)$ with finite state space S and adapted to the filtration $\mathbb{F}$;
(v) $\{\eta(t) : t \in [s,T]\}$ is a K-dimensional Lévy process defined on $(\Omega, \mathcal{F}, P)$ over $[s,T]$ and adapted to the filtration $\mathbb{F}$;
(vi) $u : [s,T] \times \Omega \to U$ is an $\{\mathcal{F}_t^s\}_{t\geq s}$-adapted process on $(\Omega, \mathcal{F}, P)$;
(vii) under $u(\cdot)$, for any $y \in \mathbb{R}^N$ and $i \in S$, the stochastic differential equation (3) admits a unique solution $X(\cdot)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t\geq s}, P)$.

We call $\mathcal{U}^w[s,T]$ the set of *weak admissible controls*. Whenever the meaning is clear from the context, we will use the shorter notation $u(\cdot) \in \mathcal{U}^w[s,T]$ for the 8-tuple $(\Omega, \mathcal{F}, \mathbb{F}, P, W(\cdot), \alpha(\cdot), \eta(\cdot), u(\cdot)) \in \mathcal{U}^w[s,T]$.

The optimal control problem under consideration can be restated in dynamic programming form as follows. For any $(s,y,i) \in [0,T) \times \mathbb{R}^N \times S$, find $\bar{u}(\cdot) \in \mathcal{U}^w[s,T]$ such that

$$J(s,y,i;\bar{u}(\cdot)) = \sup_{u(\cdot)\in\mathcal{U}^w[s,T]} J(s,y,i;u(\cdot)) \ . \quad (5)$$

Note that under assumptions (A1)-(A2), for any $(s,y,i) \in [0,T) \times \mathbb{R}^N \times S$ and $u(\cdot) \in \mathcal{U}^w[s,T]$, SDE (3) admits a unique solution $X(\cdot) = X_{s,y,i}(\cdot;u(\cdot))$ (see [1, 19, 20]), and the objective functional in Eqn. (4) is well-defined. Thus, the *value function* is well-defined by

$$\begin{cases} V(s,y,i) = \sup\limits_{u(\cdot)\in\mathcal{U}^w[s,T]} J\left(s,y,i;u(\cdot)\right), \\ V(T,y,i) = \psi(T,y,i) \qquad\qquad \left((s,y,i) \in [0,T) \times \mathbb{R}^N \times S\right). \end{cases} \quad (6)$$

In Section 3, we will state and prove a dynamic programming principle for the value function V. Furthermore, we will obtain the Hamilton-Jacobi-Bellman equation associated with the optimal control problem under consideration and the corresponding verification theorem. Section 4 contains an application of the results obtained in Section 3 to an optimal consumption-investment problem for a financial market with asset prices described by Markov-switching linear jump-diffusions.

3. Dynamic programming principle and HJB equation

The goal of this section is to obtain a dynamic programming principle for (6) and to derive the associated HJB equation. We start by providing a well-known property of the value function that will be useful in the proof of the dynamic programming principle.

Recall that the paths of the state variable component $X(\cdot)$ consist of that of an Lévy process interlaced by switches of the Markov process $\{\alpha(t) : t \in [0,T]\}$. Standard results on stochastic differential equations (see, e.g., [1]) guarantee that for any $s \in [0,T]$, $y, \hat{y} \in \mathbb{R}^N$, $i \in S$ and any w-admissible control $u(\cdot) \in \mathcal{U}^w[s,T]$, there exists $C > 0$ such that

$$E\left(\sup_{t\in[s,T]} |X_{s,y,i}(t;u(\cdot)) - X_{s,\hat{y},i}(t;u(\cdot))|^2\right) \leq C|y-\hat{y}|^2. \tag{7}$$

Combining assumptions (A1) and (A2) with the estimate in Eqn. (7), it is possible to obtain the existence of a positive constant C' such that

$$|J(s,y,i;u(\cdot)) - J(s,\hat{y},i;u(\cdot))|^2 \leq C'|y-\hat{y}|^2$$

for every $u(\cdot) \in \mathcal{U}^w[s,T]$. Taking the supremum in $u(\cdot) \in \mathcal{U}^w[s,T]$, we obtain the following property for the value function V defined in Eqn. (6).

Lemma 3.1. *Let conditions (A1)-(A2) hold. Then, there exists a constant $C' > 0$ such that for every $s \in [0,T]$, $y, \hat{y} \in \mathbb{R}^N$ and $i \in S$, we have*

$$|V(s,y,i) - V(s,\hat{y},i)|^2 \leq C'|y-\hat{y}|^2.$$

Let $\mathbb{F}^s = \{\mathcal{F}_t^s\}_{t\geq s}$ be the filtration jointly generated by the Brownian motion $W(\cdot)$, the Markov process $\alpha(\cdot)$ and the Lévy process $\eta(\cdot)$ over a time interval $[s,t]$ augmented by all the P-null sets in $\mathcal{F}$. For any $s \in [0,T)$ and any $(\Omega, \mathcal{F}, \mathbb{F}, P, W(\cdot), \alpha(\cdot), \eta(\cdot), u(\cdot)) \in \mathcal{U}^w[s,T]$, take $\hat{s} \in [s,T)$. Since for every $\hat{s} \in [s,T)$ the pair $(X_{s,y,i}(\hat{s};u(\cdot)), \alpha_{s,i}(\hat{s}))$ is $\mathcal{F}_{\hat{s}}^s$-measurable, the solutions $X_{s,y,i}(t;u(\cdot))$ and $X_{\hat{s},X_{s,y,i}(\hat{s},u(\cdot)),\alpha_{s,i}(\hat{s})}(t;u(\cdot))$ agree a.s. for every $t \in [\hat{s},T]$.

Lemma 3.2. *Let $(s,y,i) \in [0,T) \times \mathbb{R}^N \times S$ and $u(\cdot) \in \mathcal{U}^w[s,T]$. For any $\hat{s} \in [s,T)$, the following equality holds $P-$*a.s. *ω:*

$$\begin{aligned} &J(\hat{s}, X_{s,y,i}(\hat{s};u(\cdot)), \alpha_{s,i}(\hat{s});u(\cdot)) \\ &= E\Bigg[\int_{\hat{s}}^T L\left(t, X_{\hat{s},X_{s,y,i}(\hat{s};u(\cdot)),\alpha_{s,i}(\hat{s})}(t;u(\cdot)), \alpha_{\hat{s},\alpha_{s,i}(\hat{s})}(t), u(t)\right) \mathrm{d}t \\ &\qquad + \Psi\left(T, X_{\hat{s},X_{s,y,i}(\hat{s};u(\cdot)),\alpha_{s,i}(\hat{s})}(T;u(\cdot)), \alpha_{\hat{s},\alpha_{s,i}(\hat{s})}(T)\right) \;\Big|\; \mathcal{F}_{\hat{s}}^s\Bigg](\omega). \end{aligned}$$

We will now use the two lemmas above to prove a dynamic programming principle for the value function in Eqn. (6).

Theorem 3.3 (Dynamic programming principle)**.** *Assume that conditions (A1)-(A2) hold. Then, for any* $(s, y, i) \in [0, T) \times \mathbb{R}^N \times S$ *we have that*

$$V(s,y,i) = \sup_{u\in\mathcal{U}^w[s,T]} E\Bigg[\int_s^{\hat{s}} L\left(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t)\right)\mathrm{d}t + V\left(\hat{s}, X_{s,y,i}(\hat{s};u(\cdot)), \alpha_{s,i}(\hat{s})\right)\Bigg] \tag{8}$$

for all $\hat{s} \in [s, T]$.

Proof. Denote the right-hand side of (8) by $\overline{V}(s, y, i)$. Start by noting that for any $\epsilon > 0$ there exists $u(\cdot) \in \mathcal{U}^w[s, T]$ such that

$$V(s,y,i) - \epsilon < J(s,y,i;u(\cdot)).$$

Recalling that the definition of the objective functional in Eqn. (4) and letting $\hat{s} \in [s, T]$, we obtain

$$\begin{aligned} V(s,y,i) - \epsilon < E\Bigg[&\int_s^{\hat{s}} L\left(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t)\right)\mathrm{d}t \\ &+ E\Big[\int_{\hat{s}}^{T} L\left(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t)\right)\mathrm{d}t \\ &+ \Psi\left(T, X_{s,y,i}(T;u(\cdot)), \alpha_{s,i}(T)\right) \Big| \mathcal{F}_{\hat{s}}^s\Big]\Bigg]. \end{aligned}$$

Since the solutions of Eqn. (3) have the Markov property, we get

$$\begin{aligned} V(s,y,i) - \epsilon < E\Bigg[&\int_s^{\hat{s}} L\left(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t)\right)\mathrm{d}t \\ &+ E\Big[\int_{\hat{s}}^{T} L\Big(t, X_{\hat{s},X_{s,y,i}(\hat{s};u(\cdot)),\alpha_{s,i}(\hat{s})}(t;u(\cdot)), \alpha_{\hat{s},\alpha_{s,i}(\hat{s})}(t), u(t)\Big)\mathrm{d}t \\ &+ \Psi\Big(T, X_{\hat{s},X_{s,y,i}(\hat{s};u(\cdot)),\alpha_{s,i}(\hat{s})}(T;u(\cdot)), \alpha_{\hat{s},\alpha_{s,i}(\hat{s})}(T)\Big) \Big| \mathcal{F}_{\hat{s}}^s\Big]\Bigg]. \end{aligned}$$

Using the representation provided by Lemma 3.2, we obtain from the previous inequality that

$$\begin{aligned} V(s,y,i) - \epsilon < E\Bigg[&\int_s^{\hat{s}} L\left(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t)\right)\mathrm{d}t \\ &+ J(\hat{s}, X_{s,y,i}(\hat{s};u(\cdot)), \alpha_{s,i}(\hat{s});u(\cdot))\Bigg]. \end{aligned}$$

Hence, combining the definition of the value function of Eqn. (6) with the previous inequality, we get

$$V(s,y,i) - \epsilon < E\Bigg[\int_s^{\hat{s}} L\left(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t)\right)\mathrm{d}t + V\left(\hat{s}, X_{s,y,i}(\hat{s};u(\cdot)), \alpha_{s,i}(\hat{s})\right)\Bigg] \leq \overline{V}\left(s,y,i\right).$$

We will now prove the converse. Let $(s,y,i) \in [0,T) \times \mathbb{R}^n \times S$ and fix an arbitrary control $u(\cdot) \in \mathcal{U}^w[s,T]$. Using Lemma 3.1 and the comments preceding its statement, for any $\hat{s} \in [s,T]$ and $\epsilon > 0$, there exists a weak admissible control $u_\epsilon(\cdot) \in \mathcal{U}^w[\hat{s},T]$ such that

$$V(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s})) - \epsilon \leq J(\hat{s}, X_{s,y,i}(\hat{s}; u(\cdot)), \alpha_{s,i}(\hat{s}); u_\epsilon(\cdot)). \tag{9}$$

Define the process

$$\tilde{u}_\epsilon(t,\omega) = \begin{cases} u\,(t,\omega) & \text{if } t \in [s,\hat{s}), \\ u_\epsilon\,(t,\omega) & \text{if } t \in [\hat{s},T]. \end{cases}$$

Using the measurable selection theorem it is possible to guarantee that $\tilde{u}_\epsilon(\cdot)$ is progressively measurable, and thus $\tilde{u}_\epsilon(\cdot) \in \mathcal{U}^w[s,T]$.

¿From the definition of the value function and the weak admissible control $\tilde{u}_\epsilon(\cdot)$, we get

$$\begin{aligned} V(s,y,i) &\geq J(s,y,i;\tilde{u}_\epsilon(\cdot)) \\ &= E\Bigg[\int_s^{\hat{s}} L\,(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t))\,\mathrm{d}t \\ &\quad + E\Bigg[\int_{\hat{s}}^{T} L\left(t, X_{\hat{s},X_{s,y,i}(\hat{s};u(\cdot)),\alpha_{s,i}(\hat{s})}(t;u_\epsilon(\cdot)), \alpha_{\hat{s},\alpha_{s,i}(\hat{s})}(t), u_\epsilon(t)\right)\mathrm{d}t \\ &\qquad + \Psi\left(T, X_{\hat{s},X_{s,y,i}(\hat{s};u(\cdot)),\alpha_{s,i}(\hat{s})}(T;u_\epsilon(\cdot)), \alpha_{\hat{s},\alpha_{s,i}(\hat{s})}(T)\right) \;\Big|\; \mathcal{F}_{\hat{s}}^s\Bigg]\Bigg]. \end{aligned}$$

Combining Lemma 3.2 with the previous inequality, we obtain

$$\begin{aligned} V(s,y,i) \geq E\Bigg[&\int_s^{\hat{s}} L\,(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t))\,\mathrm{d}t \\ &+ J(\hat{s}, X_{s,y,i}(\hat{s};u(\cdot)), \alpha_{s,i}(\hat{s}); u_\epsilon(\cdot))\Bigg]. \end{aligned}$$

Hence, inequality (9) implies that

$$\begin{aligned} V(s,y,i) \geq E\Bigg[&\int_s^{\hat{s}} L\,(t, X_{s,y,i}(t;u(\cdot)), \alpha_{s,i}(t), u(t))\,\mathrm{d}t \\ &+ V(\hat{s}, X_{s,y,i}(\hat{s};u(\cdot)), \alpha_{s,i}(\hat{s})) - \epsilon\Bigg]. \end{aligned}$$

The proof is completed by taking the supremum over $u(\cdot) \in \mathcal{U}^w[s,T]$. □

Proposition 3.4. *Assume that conditions (A1)-(A2) hold. If the triple* $(\bar{X}(\cdot), \bar{\alpha}(\cdot), \bar{u}(\cdot))$ *is optimal for problem of Eqn.* (5)*, then*

$$\begin{aligned} V(t, \bar{X}(t), \bar{\alpha}(t)) = E\Bigg[&\int_t^T L\left(r, \bar{X}(r), \bar{u}(r), \bar{\alpha}(r)\right)\mathrm{d}r \\ &+ \Psi\left(T, \bar{X}(T), \bar{\alpha}(T)\right) \;\Big|\; \mathcal{F}_t^s\Bigg] \qquad P-\text{a.s.} \end{aligned}$$

for every $t \in [s,T]$.

Proof. Let $(\bar{X}(\cdot), \bar{u}(\cdot), \bar{\alpha}(\cdot))$ be an optimal solution for problem of Eqn. (5). Then, we have that

$$V(s, y, i) = J(s, y, i; \bar{u}(\cdot)).$$

Letting $t \in [s, T]$ and using Lemma 3.2, we obtain

$$J(s, y, i; \bar{u}(\cdot)) = E\left[\int_s^t L\left(r, \bar{X}(r), \bar{\alpha}(r), \bar{u}(r)\right) \mathrm{d}r + J(t, \bar{X}(t), \bar{\alpha}(t); \bar{u}(t))\right].$$

Using the dynamic programming principle, Theorem 3.3, we obtain the following sequence of inequalities:

$$\begin{aligned} V(s, y, i) &\leq E\left[\int_s^t L\left(r, \bar{X}(r), \bar{\alpha}(r), \bar{u}(r)\right) \mathrm{d}r\right] + E\left[J(t, \bar{X}(t), \bar{\alpha}(t); \bar{u}(t))\right] \\ &\leq E\left[\int_s^t L\left(r, \bar{X}(r), \bar{\alpha}(r), \bar{u}(r)\right) \mathrm{d}r\right] + E\left[V(t, \bar{X}(t), \bar{\alpha}(t))\right] \\ &\leq V(s, y, i). \end{aligned}$$

Thus, all the inequalities above are indeed equalities. Therefore, we obtain that

$$E[J(t, \bar{X}(t), \bar{\alpha}(t); \bar{u}(t))] = E[V(t, \bar{X}(t), \bar{\alpha}(t))].$$

The result then follows by observing that

$$V(t, \bar{X}(t), \bar{\alpha}(t)) \geq J(t, \bar{X}(t), \bar{\alpha}(t); \bar{u}(t)) \quad P - \text{a.s.},$$

completing the proof. □

We will now use the dynamic programming principle to obtain the corresponding HJB equation, a sequence of partial integro-differential equation, indexed by the state of the Markov process $\alpha(\cdot)$, whose "solution" is the value function of the optimal control problem under consideration here. Let $I \subseteq \mathbb{R}$ be an interval and denote by $C^{1,2}\left(I \times \mathbb{R}^N; \mathbb{R}\right)$ the set of all continuous functions $V : I \times \mathbb{R}^N \to \mathbb{R}$ such that V_t, V_X, and V_{XX} are all continuous functions of $(t, x) \in I \times \mathbb{R}^N$. Moreover, let $\mathrm{tr}(A)$ denote the trace of a $N \times N$ symmetric real matrix A and let $\langle \cdot, \cdot \rangle$ denote the inner product in $\mathbb{R}^n$.

Theorem 3.5. *Suppose that conditions (A1)-(A2) hold and that the value function V is such that $V(\cdot, \cdot, \alpha) \in C^{1,2}\left([0, T) \times \mathbb{R}^N; \mathbb{R}\right)$ for every $\alpha \in S$. Then, for each $\alpha \in S$, the value function $V(\cdot, \cdot, \alpha)$ defined on $[0, T) \times \mathbb{R}^N$ is the solution of the Hamilton-Jacobi-Bellman equation:*

$$\begin{cases} V_t + \sup\limits_{u \in U} \mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX}) = 0, \\ V(T, X, \alpha) = \Psi(T, X, \alpha) \qquad \left((t, X, \alpha) \in [0, T) \times \mathbb{R}^N \times S\right), \end{cases} \tag{10}$$

where the Hamiltonian function $\mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX})$ *is defined by*

$$\begin{aligned}\mathcal{H}(t, X, \alpha, u, V, V_X, V_{XX}) &= L(t, X, \alpha, u) + \langle V_X(t, X, \alpha), f(t, X, \alpha, u)\rangle \\ &+ \frac{1}{2}\mathrm{tr}\left(g^T(t, X, \alpha, u)V_{XX}(t, X, \alpha)g(t, X, \alpha, u)\right) \\ &+ \sum_{j\in S: j\neq\alpha} q_{\alpha j}\left(V(t, X, j) - V(t, X, \alpha)\right) \\ &+ \sum_{k=1}^{K}\int_{\mathbb{R}_0^1} W_k(t, X, \alpha, u, V, V_X, z_k)\nu_k(\mathrm{d}z_k)\end{aligned}$$

and the auxiliary functions $W_k(t, X, \alpha, u, V, V_X, z_k)$, $k = 1, \ldots, K$, *are defined by*

$$\begin{aligned}W_k(t, X, \alpha, u, V, V_X, z_k) &= V(t, X + h^{(k)}(t, X, \alpha, u, z_k), \alpha) - V(t, X, \alpha) \\ &\quad -\langle V_X(t, X, \alpha), h^{(k)}(t, X, \alpha, u, z_k)\rangle.\end{aligned}$$

Proof. Fix $(s, y, i) \in [0, T)\times\mathbb{R}^N \times S$ and $u \in U$. Let $(X(\cdot), \alpha(\cdot))$ be the state trajectory associated with the control $u(\cdot) \in \mathcal{U}^w[s, T]$, where $u(t)$ is constant such that $u(t) \equiv u$. Take $\hat{s} \in [s, T]$. Using Itô's formula (see Lemma A.1 in Appendix A), we obtain

$$\begin{aligned}V(\hat{s}, x(\hat{s}), \alpha(\hat{s})) - V(s, y, i) &= \int_s^{\hat{s}} a\left(t, X(t), \alpha(t), u(t)\right)\mathrm{d}t \\ &+ \int_s^{\hat{s}} b\left(t, X(t), \alpha(t), u(t)\right)\mathrm{d}W(t) + \int_s^{\hat{s}} c\left(t, X(t), \alpha(t), u(t)\right)\mathrm{d}M(t) \\ &+ \int_s^{\hat{s}}\sum_{k=1}^{K}\int_{\mathbb{R}_0^1} d_k\left(t, X(t), \alpha(t), u(t), z_k\right)\tilde{J}_k(\mathrm{d}t, \mathrm{d}z_k), \qquad (11)\end{aligned}$$

where

$$\begin{aligned}a(t, X, \alpha, u) &= V_t(t, X, \alpha) + \langle V_X(t, X, \alpha), f(t, X, \alpha, u)\rangle \\ &+ \frac{1}{2}\mathrm{tr}\left(g^T(t, X, \alpha, u)V_{XX}(t, X, \alpha)g(t, X, \alpha, u)\right) \\ &+ \sum_{j\in S: j\neq\alpha} q_{\alpha j}\left(V(t, X, j) - V(t, X, \alpha)\right) \qquad (12) \\ &+ \sum_{k=1}^{K}\int_{\mathbb{R}_0^1} W_k(t, X, \alpha, u, V, V_X, z_k)\nu_k(\mathrm{d}z_k)\end{aligned}$$

and

$$\begin{aligned}b(t, X, \alpha, u) &= (V_X(t, X, \alpha))^T g(t, X, \alpha, u) \\ c(t, X, \alpha, u) &= \sum_{j\in S: j\neq\alpha}\left(V(t, X, j) - V(t, X, \alpha)\right) \\ d_k(t, X, \alpha, u, z_k) &= V(t, X + h^{(k)}(t, X, \alpha, u, z_k), \alpha) - V(t, X, \alpha).\end{aligned}$$

Since $W(\cdot)$, $M(\cdot)$ and $\tilde{J}(\cdot, \mathrm{d}z)$ are all martingales, the expected value of the stochastic integrals against $\mathrm{d}W$, $\mathrm{d}M$, and $\tilde{J}$ all vanish. Using the observation

above, dividing the identity of Eqn. (11) by $\hat{s}-s$, where $s \neq \hat{s}$, and taking the expectation in both sides, we obtain

$$\frac{E\left[V(\hat{s}, X(\hat{s}), \alpha(\hat{s})) - V(s,y,i)\right]}{\hat{s}-s} = \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} a(t, X(t), \alpha(t), u(t)) \mathrm{d}t\right]. \tag{13}$$

Using Theorem 3.3, we get that

$$-\frac{E\left[V(\hat{s}, X(\hat{s}), \alpha(\hat{s})) - V(s,y,i)\right]}{\hat{s}-s} \geq \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} L\left(t, X(t), \alpha(t), u(t)\right) \mathrm{d}t\right].$$

Combining identity of Eqn. (13) with the last inequality, we obtain

$$\frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} L\left(t, X(t), \alpha(t), u(t)\right) + a(t, X(t), \alpha(t), u(t)) \, \mathrm{d}t\right] \leq 0.$$

Letting $\hat{s} \downarrow s$, we conclude that

$$V_t(s,y,i) + \mathcal{H}\left(s, y, i, u, V(s,y,i), V_X(s,y,i), V_{XX}(s,y,i)\right) \leq 0$$

for every $u \in U$. This results in

$$V_t(s,y,i) + \sup_{u \in U} \mathcal{H}\left(s, y, i, u, V(s,y,i), V_X(s,y,i), V_{XX}(s,y,i)\right) \leq 0.$$

On the other hand, for any $\epsilon > 0$ and any $\hat{s} \in (s, T]$ with $\hat{s} - s$ small enough, there exists a $\tilde{u}(\cdot) := u_{\epsilon,\hat{s}}(\cdot) \in \mathcal{U}^w[s,T]$ such that

$$V(s,y,i) - \epsilon(\hat{s}-s) \leq E\left[\int_s^{\hat{s}} L\left(t, X(t), \alpha(t), \tilde{u}(t)\right) \mathrm{d}t + V\left(\hat{s}, X(\hat{s}), \alpha(\hat{s})\right)\right].$$

Rearranging terms in the previous inequality, we get

$$\epsilon \geq -\frac{E\left[V(\hat{s}, X(\hat{s}), \alpha(\hat{s})) - V(s,y,i)\right]}{\hat{s}-s} - \frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} L\left(t, X(t), \alpha(t), \tilde{u}(t)\right) \mathrm{d}t\right].$$

Combining identity of Eqn. (13) with the previous inequality, we obtain

$$\epsilon \geq -\frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} a(t, X(t), \alpha(t), \tilde{u}(t)) \mathrm{d}t + \int_s^{\hat{s}} L\left(t, X(t), \alpha(t), \tilde{u}(t)\right) \mathrm{d}t\right].$$

Therefore, we conclude that

$$\epsilon \;\; \geq \;\; -\frac{1}{\hat{s}-s} E\left[\int_s^{\hat{s}} V_t\left(t, X(t), \alpha(t)\right) + \widetilde{\mathcal{H}}\left(t, X(t), \alpha(t), \tilde{u}(t)\right) \, \mathrm{d}t\right],$$

where $\widetilde{\mathcal{H}}\left(t, X(t), \alpha(t), \tilde{u}(t)\right)$ is notation for

$$\mathcal{H}\left(t, X(t), \alpha(t), \tilde{u}(t), V(t, X(t), \alpha(t)), V_X(t, X(t), \alpha(t)), V_{XX}(t, X(t), \alpha(t))\right).$$

Using the uniform continuity in assumption $(A2)$, we obtain

$$-\epsilon \leq V_t(s,y,i) + \sup_{u \in U} \mathcal{H}\left(s, y, i, u, V(s,y,i), V_X(s,y,i), V_{XX}(s,y,i)\right),$$

completing the proof. □

Before proceeding to the statement and proof of the next result, let us introduce the notation $\widetilde{H}(s,y,i,u)$ for the function

$$\widetilde{\mathcal{H}}(s,y,i,u) = \mathcal{H}(s,y,i,u,V(s,y,i),V_X(s,y,i),V_{XX}(s,y,i)).$$

We will now state the verification theorem associated with the dynamic programming principle, Theorem 3.3, and the corresponding HJB-equation (10).

Proposition 3.6 (Verification Theorem)**.** *Assume that conditions (A1)-(A2) hold and that $V(\cdot,\cdot,a) \in C^{1,2}([0,T]\times\mathbb{R}^N;\mathbb{R})$ for each $a \in S$. Let $V(s,y,i)$ be a solution of the HJB equation* (10)*. Then, the inequality*

$$V(s,y,i) \geq J(s,y,i;u(\cdot))$$

holds for every $u(\cdot) \in \mathcal{U}^w[s,T]$ and $(s,y,i) \in [0,T)\times\mathbb{R}^N\times S$. Furthermore, an admissible pair $(\bar{X}(\cdot),\bar{\alpha}(\cdot),\bar{u}(\cdot))$ is optimal for (5) *if and only if the equality*

$$V_t(t,\bar{X}(t),\bar{\alpha}(t)) + \widetilde{\mathcal{H}}(t,\bar{X}(t),\bar{\alpha}(t),\bar{u}(t)) = 0 \tag{14}$$

*holds for a.e. $t \in [s,T]$ and $\mathbb{P}-$*a.s.*.*

Proof. For any $u(\cdot) \in \mathcal{U}^w[s,T]$ and corresponding state trajectory $(X(\cdot),\alpha(\cdot))$, using Itô's formula we get

$$V(s,y,i) = E\bigg[\Psi(T,X(T),\alpha(T)) - \int_s^T a(t,X(t),\alpha(t),u(t))\mathrm{d}t\bigg],$$

where $a(t,X,\alpha,u)$ is as given in (12). Using (4) and the definition of the Hamiltonian function in the statement of Theorem 3.5, the last equality may be written as

$$\begin{aligned} V(s,y,i) &= J(s,y,i;u(\cdot)) \\ &\quad -E\bigg[\int_s^T V_t(t,X(t),\alpha(t)) + \widetilde{\mathcal{H}}(t,X(t),\alpha(t),u(t))\,\mathrm{d}t\bigg]. \end{aligned} \tag{15}$$

Using the HJB equation (10), we conclude that

$$V(s,y,i) \geq J(s,y,i;u(\cdot)),$$

completing the proof of the first part of the theorem.

To prove the second part of the theorem, let $(\bar{X}(\cdot),\bar{u}(\cdot),\bar{\alpha}(\cdot))$ be an optimal solution for (5). Applying equality (15) to $(\bar{X}(\cdot),\bar{u}(\cdot),\bar{\alpha}(\cdot))$, we get

$$\begin{aligned} V(s,y,i) &\geq J(s,y,i;\bar{u}(\cdot)) \\ &\quad -E\bigg[\int_s^T V_t(t,\bar{X}(t),\bar{\alpha}(t)) + \widetilde{\mathcal{H}}(t,\bar{X}(t),\bar{\alpha}(t),\bar{u}(t))\mathrm{d}t\bigg]. \end{aligned}$$

The desired result follows immediately from the fact that

$$V_t(t,\bar{X}(t),\bar{\alpha}(t)) + \widetilde{\mathcal{H}}(t,\bar{X}(t),\bar{\alpha}(t),\bar{u}(t)) \leq 0,$$

due to the HJB equation (10). □

4. Application to consumption-investment problems

We will now discuss the application of the results in the previous section to the topic of optimal consumption-investment problems in Markov-switching jump-diffusion financial markets.

The Brownian motion can be interpreted as small random shocks that influence the market dynamics. The jump process models large changes in the asset price, which can be interpreted as a consequence of abrupt market events or news with large impact. Finally, a transition of the Markov process to a different state models a shift in the financial market behaviour, e.g., a shift from a "bull" day to a "bear" day, or a shift from a "volatile" day to a "choppy" trading day. Similarly, for a longer term point of view, the shifts in the Markov process may be seen as modelling global changes in market behaviours such as bull or bear market periods, as well as range-bound markets.

4.1. **Problem formulation.**

Consider the setup introduced in Section 2. Namely, let $T > 0$ be a deterministic finite horizon and let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space. Define on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ a standard one-dimensional Brownian motion $W(t) = \{W(t) : t \in [0,T]\}$, a continuous time Markov process $\{\alpha(t) : t \in [0,T]\}$ with a finite space state $S = \{a_1, \ldots, a_n\}$ and a generator $Q = (q_{ij})_{i,j \in S}$, and a one-dimensional Lévy process $\{\eta(t) : t \in [0,T]\}$ with Poisson random measure $J(t,A)$ with intensity $\nu(A) = E[J(1,A)]$. Throughout this section, we assume that the three stochastic processes, $\alpha(\cdot)$, $W(\cdot)$ and $\eta(\cdot)$, are independent.

Using the stochastic processes described above, we define a continuous-time financial market consisting of one risk-free asset and one risky-asset. More precisely, we assume that the prices of the risk-free asset $\{S_0(t) : t \in [0,T]\}$ and the risky asset $\{S_1(t) : t \in [0,T]\}$ evolve according to the stochastic differential equations

$$\begin{aligned} \mathrm{d}S_0(t) &= r(t,\alpha(t_-))S_0(t_-)\mathrm{d}t, \\ \mathrm{d}S_1(t) &= \mu(t,\alpha(t_-))S_1(t_-)\mathrm{d}t + \sigma(t,\alpha(t_-))S_1(t_-)\mathrm{d}W(t) \\ &\quad + S_1(t_-)\int_{\mathbb{R}_0} h(t,\alpha(t_-),z)\tilde{J}(\mathrm{d}t,\mathrm{d}z) \end{aligned} \tag{16}$$

with positive initial conditions $S_0(0) = s_0$ and $S_1(0) = s_1$. Note that the financial market coefficients depend both on time and the state of the Markov process $\{\alpha(t) : t \in [0,T]\}$. More precisely, we assume that the riskless interest rate $r(t,a)$, the risky-asset appreciation rate $\mu(t,a)$ and the risky-asset volatility $\sigma(t,a)$, are deterministic continuous functions on the interval $[0,T]$ for every fixed $a \in S$, and that $h(t,a,z)$ is a deterministic continuous function of $t \in [0,T]$ for every $z \in \mathbb{R}_0^1$ and $a \in S$. Additionally, we assume that the risk-free interest rate $r(t,a)$ is positive for every $(t,a) \in [0,T] \times S$.

In order to ensure that the risky asset price $S_1(t)$ remains positive for every $t \in [0,T]$, we impose that

$$h(t,a,z) > -1,$$

for every $t \in [0,T]$ and $a \in S$. Finally, we also assume that

$$E\left[\int_0^T \Big(|\sigma(t,a)|^2 + \int_{\mathbb{R}_0^1} |h(t,a,z)|^2 \nu(\mathrm{d}z)\Big)\mathrm{d}t\right] < \infty$$

for every $a \in S$.

We now introduce the control variables. The *consumption process* $\{c(t) : t \in [0,T]\}$ is a $(\mathcal{F}_t)_{0\le t\le T}$-progressively measurable non-negative process satisfying the following integrability condition for the investment horizon $T > 0$:

$$\int_0^T c(t)\ \mathrm{d}t < \infty \qquad \text{a.s..}$$

Let $\theta(t)$ denote the fraction of the agent's wealth allocated to the risky asset S_1 at time $t \in [0,T]$. We assume that $\{\theta(t) : t \in [0,T]\}$ is $(\mathcal{F}_t)_{0\le t\le T}$-progressively measurable and that, for the fixed maximum investment horizon $T > 0$, we have that

$$\int_0^T |\theta(t)|^2\ \mathrm{d}t < \infty \qquad \text{a.s..}$$

Clearly, the agent invests $1-\theta(t)$ of her wealth on the risk-free asset S_0. We now define the *wealth process* $X(t)$, for $t \in [0,T]$, through the stochastic differential equation

$$\begin{aligned}\mathrm{d}X(t) = &\left(-c(t) + \Big((1-\theta(t))r(t,\alpha(t_-)) + \theta(t)\mu(t,\alpha(t_-))\Big)X(t_-)\right)\mathrm{d}t \\ &+ \theta(t)X(t_-)\left(\sigma(t,\alpha(t_-))\mathrm{d}W(t) + \int_{\mathbb{R}_0^1} h(t,\alpha(t_-),z)\tilde{J}(\mathrm{d}t,\mathrm{d}z)\right) \quad (17)\end{aligned}$$

with initial conditions $X(0) = x$ and $\alpha(0) = a$ representing, respectively, the initial wealth and the initial state of the Markov process $\alpha(\cdot)$.

Let us denote by $\mathcal{A}(x,a)$ the set of all admissible decision strategies, i.e. all admissible choices for the control variables $(c,\theta) \in [0,+\infty) \times [0,1]$ such that the wealth process defined by (17) is square integrable with respect to $\mathrm{d}t \times \mathrm{d}P$ over $[0,T] \times \Omega$. The dependence of $\mathcal{A}(x,a)$ on $(x,a) \in \mathbb{R} \times S$ denotes the restriction imposed on the wealth process by the boundary conditions $X(0) = x$ and $\alpha(0) = a$. Similarly, let us denote by $\mathcal{A}(t,x,a)$ the set of all admissible decision strategies $(c,\theta) \in [0,+\infty) \times [0,1]$ for the dynamics of the wealth process of Eqn. (17) with boundary condition $X(t) = x$ and $\alpha(t) = a$.

The consumption-investment problem is to find consumption and investment strategies, $(c,\theta) \in \mathcal{A}(x,a)$ which maximize the expected utility

$$\begin{aligned}J(x,a;c(\cdot),\theta(\cdot)) = E\Bigg[&\int_0^T U(t,c(t),\alpha_{0,a}(t))\mathrm{d}t \\ &+\Psi(T, X_{0,x,a}(T;c(\cdot),\theta(\cdot)),\alpha_{0,a}(T))\Bigg].\end{aligned}$$

Here, $U(t,c,a)$ is the utility obtained from a consumption level $c \in [0,\infty)$ at time t when the state of the Markov process $\alpha(\cdot)$ is a, $\Psi(T,x,a)$ is the utility obtained from holding wealth x at time T when the state of the Markov

process $\alpha(\cdot)$ is a, and $(X_{s,y,i}(t;c(\cdot),\theta(\cdot)),\alpha_{s,i}(t)) \in \mathbb{R}\times S$ is the solution of Eqn. (17) associated with the strategies $c(\cdot),\theta(\cdot)$ and initial condition from (x,a) when $t=0$.

Proceeding as described in Section 2, we introduce the expected utility

$$J(s,y,i;c(\cdot),\theta(\cdot)) = E\Bigg[\int_s^T U(t,c(t),\alpha_{s,i}(t))\mathrm{d}t \\ +\Psi(T,X_{s,y,i}(T;c(\cdot),\theta(\cdot)),\alpha_{s,i}(T))\Bigg], \quad (18)$$

where $(X_{s,y,i}(t;c(\cdot),\theta(\cdot)),\alpha_{s,i}(t)) \in \mathbb{R}\times S$ is the solution of Eqn. (17) associated with the strategies $c(\cdot),\theta(\cdot)$ and starting from (y,i) when $t=s$. Using dynamic programming techniques it is possible to obtain a rather complete description for the behaviour of the maximum expected utility, or value function, given by

$$\begin{cases} V(t,x,a) = \sup\limits_{(c,\theta)\in\mathcal{A}(t,x,a)} J(t,x,a;c(\cdot),\theta(\cdot)), \\ V(T,x,a) = \psi(T,x,a) \qquad\qquad ((t,x,a)\in[0,T)\times\mathbb{R}\times S)\,. \end{cases}$$

Indeed, in the next sections we will use the theory built for the proof of Theorems 3.3 and 3.5 to study the optimal strategies for the expected utility in Eqn. (18) both in the case where the utility functions are of power type and in the case of logarithmic utilities.

4.2. **The case of power utility functions.** In this subsection we assume that the utility functions belong to the following class of power utilities. Let

$$U(t,c,a) = \mathrm{e}^{-\rho t}\frac{c^{\gamma_a}}{\gamma_a}, \qquad \Psi(t,x,a) = \mathrm{e}^{-\rho t}\frac{x^{\gamma_a}}{\gamma_a}, \quad (19)$$

where $\gamma_a \in (0,1)$ is the risk aversion coefficient associated with the state of the Markov process $\alpha(t)=a\in S$, and $\rho>0$ is the discount rate.

In the next theorem, we will compute the optimal strategies for the class of discounted utility functions of Eqn. (19). Before providing the precise statement, let us introduce the function $F:[0,1]\times[0,T]\times S\to\mathbb{R}$ given by

$$\begin{aligned} F(\theta;t,a) &= \gamma_a\left[r(t,a)+\theta\big(\mu(t,a)-r(t,a)\big)-\frac{1}{2}(1-\gamma_a)\theta^2\sigma^2(t,a)\right] \\ &\quad +\int_{\mathbb{R}_0^1}\Big((1+\theta h(t,a,z))^{\gamma_a}-1-\gamma_a\theta h(t,a,z)\Big)\nu(\mathrm{d}z) \quad (20) \end{aligned}$$

and note that

$$\begin{aligned} F'(\theta;t,a) &= \gamma_a\Big[\mu(t,a)-r(t,a)-(1-\gamma_a)\theta\sigma^2(t,a) \\ &\quad +\int_{\mathbb{R}_0^1}\big((1+\theta h(t,a,z))^{\gamma_a-1}-1\big)h(t,a,z)\nu(\mathrm{d}z)\Big], \end{aligned}$$

where the derivative is taken with respect to θ.

Theorem 4.1. *The maximum expected utility associated with Eqn.* (18) *and the discounted utility functions of Eqn.*(19) *is given by*

$$V(t,x,a) = \xi_a(t)\frac{x^{\gamma_a}}{\gamma_a}, \quad (21)$$

the corresponding optimal strategies are of the form

$$c^*(t,x,a) = x(\mathrm{e}^{\rho t}\xi_a(t))^{-1/(1-\gamma_a)}$$

and

$$\theta^*(t,a) = \begin{cases} 1, & \text{if} \quad \mu(t,a) > r(t,a) \quad \text{and} \quad F'(1;t,a) \geq 0, \\ \hat{\theta}(t,a), & \text{if} \quad \mu(t,a) > r(t,a) \quad \text{and} \quad F'(1;t,a) < 0, \\ 0, & \text{if} \quad \mu(t,a) \leq r(t,a), \end{cases}$$

where $\hat{\theta}(t,a)$ is the unique solution of $F'(\theta;t,a) = 0$ in $(0,1)$ and $\xi_a(t)$, $a \in S$, are the solutions of the following coupled ordinary differential equations terminal value problem

$$\begin{aligned} \xi_a'(t) \quad + \quad & (1-\gamma_a)\mathrm{e}^{-\rho t/(1-\gamma_a)}\xi_a(t)^{-\gamma_a/(1-\gamma_a)} + F(\theta^*(t,a);t,a)\xi_a(t) \\ & + \sum_{j\in S: j\neq a} q_{aj}(\xi_j(t) - \xi_a(t)) = 0, \qquad (22) \\ \xi_a(T) \quad = \quad & e^{-\rho T}. \end{aligned}$$

Proof. Assume for the time being that the conditions of Theorem 3.5 hold. The Hamiltonian function $\mathcal{H}$ associated with the expected utility (18) and the discounted utility functions (19) is defined by

$$\begin{aligned} &\mathcal{H}(t,x,a,c,\theta,V,V_x,V_{xx}) = \\ &\quad \mathrm{e}^{-\rho t}\frac{c^{\gamma_a}}{\gamma_a} + \bigg(-c + \Big(r(t,a) + \theta\big(\mu(t,a) - r(t,a)\big)\Big)x\bigg)V_x(t,x,a) \\ &\quad + \frac{x^2}{2}\big(\theta\sigma(t,a)\big)^2 V_{xx}(t,x,a) + \sum_{j\in S: j\neq a} q_{aj}(V(t,x,j) - V(t,x,a)) \\ &\quad + \int_{\mathbb{R}_0^1} \Big(V(t,x+\theta x h(t,a,z)) - V(t,x,a) - \theta x V_x(t,x,a)h(t,a,z)\Big)\nu(\mathrm{d}z) \end{aligned}$$

and the Hamilton-Jacobi-Bellman equation is defined by

$$V_t + \sup_{(c,\theta)\in[0,\infty)\times[0,1]} \mathcal{H}(t,x,a,c,\theta,V,V_x,V_{xx}) = 0\ .$$

Considering an ansatz of the form of Eqn. (21) and substituting in the HJB equation above, we get

$$\begin{aligned} \xi_a'(t)\frac{x^{\gamma_a}}{\gamma_a} + \sup_{(c,\theta)\in[0,\infty)\times[0,1]} \Bigg\{ & \mathrm{e}^{-\rho t}\frac{c^{\gamma_a}}{\gamma_a} \\ & + \bigg(-c + \Big(r(t,a) + \theta(\mu(t,a) - r(t,a))\Big)x\bigg)\xi_a(t)x^{\gamma_a - 1} \qquad (23) \\ & + \frac{\theta^2}{2}\sigma^2(t,a)(\gamma_a - 1)\xi_a(t)x^{\gamma_a} + \frac{x^{\gamma_a}}{\gamma_a}\sum_{j\in S: j\neq a} q_{aj}\big(\xi_j(t) - \xi_a(t)\big) \\ & + \xi_a(t)\frac{x^{\gamma_a}}{\gamma_a}\int_{\mathbb{R}_0^1}\Big((1+\theta h(t,a,z))^{\gamma_a} - 1 - \gamma_a\theta h(t,a,z)\Big)\nu(\mathrm{d}z)\Bigg\} = 0. \end{aligned}$$

Note that the optimization problem in Eqn. (23) breaks down into two independent optimization problems and its solution can be obtained in a

sequential way. We start by optimizing the Eqn. (23) with respect to c, before proceeding to optimize with respect to the variable θ.

Since $\gamma_a \in (0,1)$ for every $a \in S$, the quantity to be maximized in (23) is strictly concave with respect to the control variable c. Indeed, the first order condition associated with the optimization problem above provides a maximizer $c^*(t,x,a)$, which is given by

$$c^*(t,x,a) = x(\mathrm{e}^{\rho t}\xi_a(t))^{-1/(1-\gamma_a)}.$$

Replacing c by $c^*(t,x,a)$ in Eqn. (23) and factoring out the term x^{γ_a}/γ_a, we obtain that

$$\xi_a'(t) + \sup_{\theta\in[0,1]} \Bigg\{ (1-\gamma_a)e^{-\rho t/(1-\gamma_a)}\xi_a(t)^{-\gamma_a/(1-\gamma_a)} + F(\theta;t,a)\xi_a(t) + \sum_{j\in S: j\neq a} q_{aj}\big(\xi_j(t) - \xi_a(t)\big) \Bigg\} = 0, \quad (24)$$

where $F(\theta;t,a)$ is as introduced in Eqn. (20). Note that the first order condition with respect to θ is just $F'(\theta;t,a) = 0$ and that since $0 < \gamma_a < 1$, the second derivative of $F(\theta;t,a)$ with respect to θ, represented by

$$F''(\theta;t,a) = -\gamma_a(1-\gamma_a)\Big[\sigma^2 + \int_{\mathbb{R}_0^1} (1+\theta h(t,a,z))^{\gamma_a - 2} h^2(t,a,z)\nu(\mathrm{d}z)\Big],$$

is negative for every $\theta \in [0,1]$. Taking into account the constraint $\theta \in [0,1]$ and the concavity of $F(\theta;t,a)$, we conclude that the maximization problem in (24) has a unique solution $\theta^*(t,a)$. Moreover, from the definition of the function $F(\theta;t,a)$, it is possible to check that

i) if $\mu(t,a) - r(t,a) > 0$ and $F(1;t,a) < 0$, then there exists $\hat{\theta}(t,a) \in (0,1)$ such that $F'(\hat{\theta}(t,a);t,a) = 0$ and, consequently, $\theta^*(t,a) = \hat{\theta}(t,a)$;
ii) if $\mu(t,a) - r(t,a) > 0$ and $F(1;t,a) \geq 0$, then $\theta^*(t,a) = 1$;
iii) if $\mu(t,a) - r(t,a) \leq 0$, then $\theta^*(t,a) = 0$.

We will now check that the system of ordinary differential equations in Eqn. (22) admits a unique positive solution. For that purpose, we consider the new variable

$$z_a(t) = \big(e^{\rho t}\xi_a(t)\big)^{1/(1-\gamma_a)} : \quad (a \in S).$$

A straightforward computation shows that $z_a(t)$ must satisfy the system of ordinary differential equations

$$\begin{aligned} z_a'(t) &= \frac{\rho}{1-\gamma_a} - 1 - \frac{1}{1-\gamma_a} + F(\theta^*(t,a);t,a)z_a(t) \\ &\qquad - z_a^{\gamma_a} \sum_{j\in S: j\neq a} q_{aj}(z_j^{1-\gamma_a}(t) - z_a^{1-\gamma_a}(t)) = 0, \quad (25) \\ z_a(T) &= 1. \end{aligned}$$

A simple analysis shows that the solutions of Eqn. (25) are positive and bounded away from zero for every $t \in [0,T]$. Clearly, the same statement also holds for the solutions $\xi_a(t)$ of Eqn. (22). □

A few remarks concerning the optimal strategies determined in Theorem 4.1 are in order. We start by noting that θ^* does not depend on the wealth $x \in \mathbb{R}$, and depends on time $t \in [0,T]$ only through its coefficients dependence (i.e. for a financial market model with constant coefficients, θ^* is independent of t). However, as can be seen from its form, θ^* depends strongly on the state $a \in S$ of the Markov process state $\alpha(\cdot)$.

As noted in [11], the occurrence of jumps in the financial market leads to more conservative optimal strategies when compared with purely diffusive markets. More precisely, agents allocate a smaller proportion of their wealth to the risky asset and consume more compared to their current wealth.

In what concerns the optimal consumption c^*, it is clearly that it is increasing with wealth and that, for choices of coefficients compatible with standard financial market behaviour, c^* is increasing with time t.

4.3. **The case of logarithmic utility functions.** Consider the utility functions

$$U(t,c) = \mathrm{e}^{-\rho t}\ln c, \qquad \Psi(t,x) = \mathrm{e}^{-\rho t}\ln x, \tag{26}$$

where $\rho > 0$ is the discount rate.

In the next theorem, we will compute the optimal strategies for the class of discounted logarithmic utility functions of Eqn. (26). Before providing the precise statement, let us introduce the function $F : [0,1]\times[0,T]\times S \to \mathbb{R}$ given by

$$\begin{aligned} F(\theta;t,a) &= r(t,a) + \theta\big(\mu(t,a) - r(t,a)\big) - \frac{1}{2}\theta^2\sigma^2(t,a) \\ &\quad + \int_{\mathbb{R}_0^1}\Big(\ln(1+\theta h(t,a,z)) - \theta h(t,a,z)\Big)\nu(\mathrm{d}z) \end{aligned} \tag{27}$$

and note that

$$F'(\theta;t,a) = \mu(t,a) - r(t,a) - \theta\sigma^2(t,a) - \int_{\mathbb{R}_0^1}\frac{\theta h^2(t,a,z)}{(1+\theta h(t,a,z))}\nu(\mathrm{d}z),$$

where the derivative is taken with respect to θ.

Theorem 4.2. *The maximum expected utility associated with Eqn.* (18) *and the discounted logarithmic utility functions* (26) *is defined by*

$$V(t,x,a) = \xi(t)\ln x + \zeta_a(t), \tag{28}$$

the corresponding optimal strategies are of the form

$$c^*(t,x,a) = \frac{\mathrm{e}^{-\rho t}}{\xi(t)}x$$

and

$$\theta^*(t,a) = \begin{cases} 1, & \text{if} \quad \mu(t,a) > r(t,a) \quad \text{and} \quad F'(1;t,a) \geq 0, \\ \hat{\theta}(t,a), & \text{if} \quad \mu(t,a) > r(t,a) \quad \text{and} \quad F'(1;t,a) < 0, \\ 0, & \text{if} \quad \mu(t,a) \leq r(t,a), \end{cases}$$

where $\hat{\theta}(t,a)$ is the unique solution of $F'(\theta;t,a) = 0$ in $(0,1)$, $\xi(t)$ is determined by

$$\xi(t) = \frac{(\rho-1)}{\rho}\mathrm{e}^{-\rho T} + \frac{\mathrm{e}^{-\rho t}}{\rho}$$

and $\zeta_a(t)$, $a \in S$, are the solutions of the following coupled ordinary differential equations terminal value problem

$$\begin{aligned} \zeta_a'(t) &- \mathrm{e}^{-\rho t}\left(\rho t + \ln \xi(t) + 1\right) + F(\theta^*(t,a);t,a)\xi(t) \\ &+ \sum_{j\in S: j\neq a} q_{aj}(\zeta_j(t) - \zeta_a(t)) = 0, \\ \zeta_a(T) &= 0. \end{aligned}$$

Proof. Assume that the conditions of Theorem 3.5 hold. The Hamiltonian function $\mathcal{H}$ associated with the expected utility of Eqn. (18) and the discounted logarithmic utility functions (26) is defined by

$$\begin{aligned} &\mathcal{H}(t,x,a,c,\theta,V,V_x,V_{xx}) = \\ &\quad \mathrm{e}^{-\rho t}\ln c + \bigg(- c + \Big(r(t,a) + \theta\big(\mu(t,a) - r(t,a)\big)\Big)x\bigg)V_x(t,x,a) \\ &\quad + \frac{x^2}{2}\big(\theta\sigma(t,a)\big)^2 V_{xx}(t,x,a) + \sum_{j\in S: j\neq a} q_{aj}(V(t,x,j) - V(t,x,a)) \\ &\quad + \int_{\mathbb{R}_0^1}\Big(V(t,x+\theta x h(t,a,z)) - V(t,x,a) - \theta x V_x(t,x,a)h(t,a,z)\Big)\nu(\mathrm{d}z) \end{aligned}$$

and the Hamilton-Jacobi-Bellman equation is determined by

$$V_t + \sup_{(c,\theta)\in[0,\infty)\times[0,1]} \mathcal{H}(t,x,a,c,\theta,V,V_x,V_{xx}) = 0.$$

Considering an ansatz of the form

$$V(t,x,a) = \xi_a(t)\ln x + \zeta_a(t)$$

and substituting in the HJB equation above, we get

$$\begin{aligned} &\xi_a'(t)\ln x + \zeta_a'(t) + \sup_{(c,\theta)\in[0,\infty)\times[0,1]} \bigg\{ \mathrm{e}^{-\rho t}\ln c \\ &\quad + \bigg(- \frac{c}{x} + \Big(r(t,a) + \theta(\mu(t,a) - r(t,a))\Big)\bigg)\xi_a(t) - \frac{1}{2}(\theta\sigma(t,a))^2\xi_a(t) \\ &\quad + \ln x \sum_{j\in S: j\neq a} q_{aj}\big(\xi_j(t) - \xi_a(t)\big) + \sum_{j\in S: j\neq a} q_{aj}\big(\zeta_j(t) - \zeta_a(t)\big) \qquad (29) \\ &\quad + \int_{\mathbb{R}_0^1}\Big(\xi_a(t)\ln(1+\theta h(t,a,z)) - \xi_a(t)\theta h(t,a,z)\Big)\nu(\mathrm{d}z)\bigg\} = 0. \end{aligned}$$

We start by optimizing with respect to c, before proceeding to optimize with respect to the variable θ. The first-order condition associated with the optimization problem above provides a maximizer $c^*(t,x,a)$, given by

$$c^*(t,x,a) = \frac{\mathrm{e}^{-\rho t}}{\xi_a(t)}x.$$

Replacing c by $c^*(t,x,a)$ in Eqn. (29) we obtain that

$$\xi_a'(t)\ln x+\zeta_a'(t)+\sup_{\theta\in[0,1]}\Bigg\{\mathrm{e}^{-\rho t}\left(-\rho t+\ln x-\ln\xi_a(t)-1\right)+F(\theta;t,a)\xi_a(t)$$
$$+\ln x\sum_{j\in S:j\neq a}q_{aj}\big(\xi_j(t)-\xi_a(t)\big)+\sum_{j\in S:j\neq a}q_{aj}\big(\zeta_j(t)-\zeta_a(t)\big)\Bigg\}=0, \quad (30)$$

where $F(\theta;t,a)$ is as given in (27). Note that as long as $\xi_a(t)$ is nonzero, the first-order condition to θ is just $F'(\theta;t,a)=0$. Moreover, the second derivative of $F(\theta;t,a)$ with respect to θ, given by

$$F''(\theta;t,a)=-\sigma^2-\int_{\mathbb{R}_0^1}\frac{h^2(t,a,z)}{(1+\theta h(t,a,z))^2}\nu(\mathrm{d}z),$$

is negative for every $\theta\in[0,1]$. Taking into account the constraint $\theta\in[0,1]$ and the concavity of $F(\theta;t,a)$, we conclude that the maximization problem in Eqn. (30) has a unique solution $\theta^*(t,a)$. Moreover, from the definition of the function $F(\theta;t,a)$, it is possible to check that

i) if $\mu(t,a)-r(t,a)>0$ and $F(1;t,a)<0$, then there exists $\hat{\theta}(t,a)\in(0,1)$ such that $F'(\hat{\theta}(t,a);t,a)=0$ and, consequently, $\theta^*(t,a)=\hat{\theta}(t,a)$;
ii) if $\mu(t,a)-r(t,a)>0$ and $F(1;t,a)\geq 0$, then $\theta^*(t,a)=1$;
iii) if $\mu(t,a)-r(t,a)\leq 0$, then $\theta^*(t,a)=0$.

¿From Eqn. (30), we obtain

$$\begin{aligned}
\xi_a'(t) &+ \mathrm{e}^{-\rho t}+\sum_{j\in S:j\neq\alpha}q_{aj}(\xi_j(t)-\xi_a(t))=0,\\
\zeta_a'(t) &- \mathrm{e}^{-\rho t}\left(\rho t+\ln\xi_a(t)+1\right)+F(\theta^*(t,a);t,a)\xi_a(t)\\
&+ \sum_{j\in S:j\neq a}q_{aj}(\zeta_j(t)-\zeta_a(t))=0, \qquad (31)\\
\xi_a(T) &= \mathrm{e}^{-\rho T},\quad \zeta_a(T)=0.
\end{aligned}$$

¿From Eqn. (31) we get that $\xi_i(t)=\xi_j(t)$ for every $i,j\in S$ and every $t\in[0,T]$. Let $\xi(t)=\xi_a(t)$, $a\in S$. Then, $\xi(t)$ must be a solution of the boundary value problem

$$\begin{aligned}
\xi'(t) &+ e^{-\rho t}=0,\\
\xi(T) &= \mathrm{e}^{-\rho T},
\end{aligned}$$

which has solution

$$\xi(t)=\frac{(\rho-1)}{\rho}\mathrm{e}^{-\rho T}+\frac{\mathrm{e}^{-\rho t}}{\rho}.$$

We conclude the proof by substituting $\xi(t)$ in the equation for $\zeta_a(t)$ in Eqn. (31). Finally, we note that the ordinary differential equations describing $\zeta_a(t)$ are linear and thus have a unique solution. □

4.4. **A simple example.** We consider now, as an example, a Markov-switching jump-diffusion financial market associated with a Markov process $\{\alpha(t) : t \in [0,T]\}$ with finite space state $\mathcal{S} = \{+, 0, -\}$. We think of the state "+" as representing an upward trending market, in which case the financial markets and the underlying economy are healthy, whereas the state "0" represents a period where the financial markets are range bound, i.e. with no precise direction. On the contrary, we associate the state "−" to periods of great stress in the markets, such as a large economic or financial crisis. We restrict ourselves to the case where the financial market has constant coefficients throughout this example. Let us use the indices "+", "0" and "−" to distinguish between the market's coefficients on each state of the Markov process $\{\alpha(t) : t \in [0,T]\}$. A reasonable choice of parameters should satisfy:

1) $r_+ > r_0 > r_- \geq 0$, i.e., interest rates tend to be higher during expansion periods and lower during recessions.
2) $\mu_+ > \mu_0 > \mu_-$, i.e., risky assets appreciation rates tend to be higher during expansion periods and lower during recession periods.
3) $\mu_+ > r_+, \quad \mu_0 \approx 0, \quad \mu_- < 0$, i.e., risky asset appreciation rates tend to be larger than interest rates during expansions, negative during recessions, and close to zero for range bound markets.
4) $\sigma_+ < \sigma_0 < \sigma_-$, meaning that the degree of uncertainty associated with the financial assets return is smaller in a upward trading market when compared with a distressed market.

Moreover, it is also reasonable to assume that h_+, h_0 and h_- are such that the tail of h_- is heavier than the tail of h_0, and the tail of h_0 is heavier than the tail of h_+, i.e. it is more likely to find large jumps in distressed markets when compared to upwarding trending markets or range bound markets.

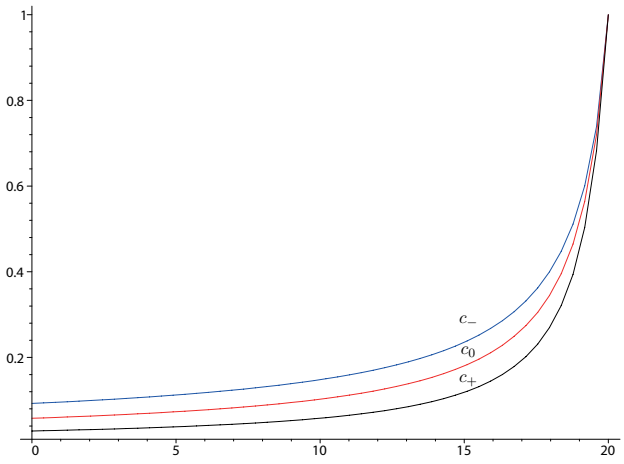


FIGURE 1. Plot of proportion of wealth consumed. The horizontal axis represents the time, and the vertical axis the consumption rate c.

The plot in Figure 1 illustrates the proportion of wealth consumed in each of one the 3 states $\mathcal{S} = \{+, 0, -\}$. We take the deterministic fixed horizon to be $T = 20$ and consider a financial market with one risk-free security

and one risky asset driven by a Markov-switching jump-diffusion market. The agent preferences are described by power utilities of the form (19) with risk aversion parameters $\gamma_+ = 0.6$, $\gamma_0 = 0.5$ and $\gamma_- = 0.4$. The discount rate is taken to be $\rho = 0.03$. We take $h(t, a, z) = z$ and $\nu(\mathrm{d}z) = \lambda \mathrm{e}^{-\lambda z}$, i.e. not depending on the state of the Markov process $\alpha(\cdot)$ for simplicity. The generator of the Markov process $\alpha(\cdot)$, $Q = (q_{ij})_{i,j\in S}$, is determined by $q_{12} = 2$, $q_{13} = 1$, $q_{21} = 2$, $q_{23} = 1$, $q_{31} = 1$ and $q_{32} = 2$. Finally we use the following choice of parameters:

(i) line c_- (in blue): risk-free interest rate is $r_- = 0.01$, the risky asset appreciation rate is $\mu_- = -0.2$ and its volatility is $\sigma_- = 0.40$;
(ii) line c_0 (in red): $r_0 = 0.03$, $\mu_0 = 0$ and $\sigma_0 = 0.21$;
(iii) line c_+ (in black): $r_+ = 0.04$, $\mu_+ = 0.21$ and $\sigma_+ = 0.13$.

Note that the consumption rate is an increasing function of time and wealth. In a growing economy, the agent has a lower risk aversion and allocates most of his wealth to the financial market. Moreover, when financial markets are not healthy, agent risk aversion increases and a larger proportion of their wealth is devoted to consumption. Although the proportion of wealth consumed is lower in upward trending markets, this does not mean that the total amount of wealth consumed is lower, since the total amount of wealth tends to increase more in upward trending markets. This strategy is optimal for a rational investor that maximizes expected utility (18), i.e., consumption in $[0, T]$ and final wealth at time at time T. Naturally, consumption decreases with an increase of risk aversion parameter γ, whereas θ^* increases with an increase of γ. Finally, θ^* also grows with risky asset appreciation rate μ and decreases with volatility coefficient σ.

5. Conclusion and Outlook

We have studied a stochastic optimal control problem for a Markov-switching jump-diffusion stochastic differential equation. We use dynamic programming techniques to derive a family of Hamilton-Jacobi-Bellman equations that describe the value function associated with the stochastic optimal control problem under consideration. We conclude with an application to a consumption-investment type problem. The present model with Markov-switching may be of great use in portfolio management, since it seems to be suitable to describe markets movement between several states (growth, crisis, range bound, bubbles). For the practical implementation of such models it would be interesting to develop statistical inference methods that would determine a reasonable number of states and detect the state of the market at a given moment in time. The development of new numerical techniques for the solution of optimal control problems of the kind studied in this paper seems to us of great interest also [22].

Appendix A. Itô's Formula

In this section we state Itô's formula for a jump-diffusion with Markov-switching. The result follows from Itô's formula for semi-martingales. See [20] for further details.

Lemma A.1 (Itô's rule for a Markov-switching jump-diffusion)**.** *Suppose that $X(t)$ is a Markov-switching jump-diffusion process given by*

$$\begin{aligned}\mathrm{d}X(t) &= f(t,X(t_-),\alpha(t_-),u(t))\mathrm{d}t+\sum_{m=1}^{M} g(t,X(t_-),\alpha(t_-),u(t))\mathrm{d}W_m(t)\\ &\quad+\sum_{k=1}^{K}\int_{\mathbb{R}_0^1} h^{(k)}(t,X(t_-),\alpha(t_-),z_k)\tilde{J}_k(\mathrm{d}t,\mathrm{d}z).\end{aligned}$$

Let $V(t,x,\alpha)$ be such that $V(\cdot,\cdot,\alpha)\in C^{1,2}([0,T]\times\mathbb{R}^N)$ for every $\alpha\in S$. Then, we have that

$$\begin{aligned}V(T,X(T),\alpha(T))-V(0,X(0),\alpha(0)) &= \int_0^T a\left(t,X(t),\alpha(t),u(t)\right)\mathrm{d}t\\ &+\int_0^T b\left(t,X(t),\alpha(t),u(t)\right)\mathrm{d}W(t)+\int_0^T c\left(t,X(t),\alpha(t),u(t)\right)\mathrm{d}M(t)\\ &+\int_0^T\sum_{k=1}^{K}\int_{\mathbb{R}_0^1} d_k\left(t,X(t),\alpha(t),u(t),z_k\right)\tilde{J}_k(\mathrm{d}t,\mathrm{d}z_k),\end{aligned}$$

where

$$\begin{aligned}a(t,X,\alpha,u) = V_t(t,X,\alpha)&+\langle V_X(t,X,\alpha),f(t,X,\alpha,u)\rangle\\ &+\frac{1}{2}tr\left(g^T(t,X,\alpha,u)V_{XX}(t,X,\alpha)g(t,X,\alpha,u)\right)\\ &+\sum_{j\in S:j\neq\alpha} q_{\alpha j}\left(V(t,X,j)-V(t,X,\alpha)\right)\\ &+\sum_{k=1}^{K}\int_{\mathbb{R}_0^1} W_k(t,X,\alpha,u,V,V_X,z_k)\nu_k(\mathrm{d}z_k)\end{aligned}$$

with

$$\begin{aligned}&W_k(t,X,\alpha,u,V,V_X,z_k)=V(t,X+h^{(k)}(t,X,\alpha,u,z_k),\alpha)-V(t,X,\alpha)\\ &-\langle V_X(t,X,\alpha),h^{(k)}(t,X,\alpha,u,z_k)\rangle\end{aligned}$$

and

$$\begin{aligned}b(t,X,\alpha,u) &= (V_X(t,X,\alpha))^T g(t,X,\alpha,u)\\ c(t,X,\alpha,u) &= \sum_{j\in S:j\neq\alpha}\left(V(t,X,j)-V(t,X,\alpha)\right)\\ d_k(t,X,\alpha,u,z_k) &= V(t,X+h^{(k)}(t,X,\alpha,u,z_k),\alpha)-V(t,X,\alpha).\end{aligned}$$

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(N. Azevedo) CEMAPRE, ISEG, Universidade Técnica de Lisboa, Lisboa, Portugal
E-mail address: nazevedo@iseg.utl.pt

(D. Pinheiro) CEMAPRE, ISEG, Universidade Técnica de Lisboa, Lisboa, Portugal
E-mail address: dpinheiro@iseg.utl.pt

(G.-W. Weber) Institute of Applied Mathematics, METU, 06531 Ankara, Turkey
E-mail address: gweber@metu.edu.tr